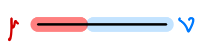
Measure Theory with Ergodic Horizons
Lecture 25

Differentiation of measures.

Let μ and ν be two measures on the same measurable space $(X, \mathcal{B})$. We will try to understand their relationship.

Recall that we say that μ is absolutely continuous wrt ν, and write $\mu \ll \nu$, if $\nu(B) = 0 \Rightarrow \mu(B) = 0$ for each $B \in \mathcal{B}$. (Also recall that when μ is finite, $\mu \ll \nu$ is equivalent to $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $\nu(B) \leq \delta \Rightarrow \mu(B) \leq \varepsilon$ for all $B \in \mathcal{B}$, justifying the term "absolute continuity".)

We now define the opposite notion to absolute continuity:

Def. Measures μ and ν on a measurable space $(X, \mathcal{B})$ are said to be orthogonal, denoted $\mu \perp \nu$, if $X = X_\mu \sqcup X_\nu$, $X_\mu, X_\nu \in \mathcal{B}$, such that $\mu(X_\nu) = 0$ and $\nu(X_\mu) = 0$. 

Such a partition is unique up to sets that are both μ and ν null.

Examples.

(a) If $\mu = \delta_{x_0}$ and ν is atomless (e.g. Lebesgue measure on $\mathbb{R}$), then $\mu \perp \nu$.

Indeed, $X_\mu := \{x_0\}$ and $X_\nu := X \setminus \{x_0\}$.

(b) Let μ_p be the Bernoulli(p) measure on $2^{\mathbb{N}}$. Let $\iota : 2^{\mathbb{N}} \hookrightarrow [0,1]$ be the usual embedding (the isomorphism between $2^{\mathbb{N}}$ and the Cantor set), i.e. $01001\ldots \mapsto 0.02002\ldots$, where the latter is the ternary representation. $\iota_* \mu_p$ lives on the standard Cantor set $C \subseteq [0,1]$, while $\lambda(C) = 0$, so $\iota_* \mu_p \perp \lambda$.

(c) Let (X, μ) be a measure space. Recall that for any non-negative measurable function $f : X \to [0, \infty]$, μ_f is the measure on X defined by $\mu_f(B) := \int_B f \, d\mu$, for all μ-meas. $B \subseteq X$. Let $X = X_0 \sqcup X_1$, with X_0, X_1 μ-measurable, and let f_0, f_1 be nonnegative measurable functions such that $f_i = \mathbb{1}_{X_i} \cdot f_i$ for $i = 0, 1$. Then $\mu_{f_0} \perp \mu_{f_1}$, witnessed by the partition $X = X_0 \sqcup X_1$.

Lebesgue decomposition theorem. Let μ and ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. Then $\mu = \mu_0 + \mu_1$ for some measures μ_0, μ_1 on $(X, \mathcal{B})$ such that $\mu_0 \ll \nu$ and $\mu_1 \perp \nu$.

Proof. It is enough to prove that $X = X_0 \sqcup X_1$, with $X_i \in \mathcal{B}$, such that $\mu|_{X_0} \ll \nu|_{X_0}$ and $\nu(X_1) = 0$. To show this, we may assume WLOG that μ and ν are finite, and build X_1 by a $\frac{1}{2}$-measure exhaustion argument (as in the proof of Sierpinski's theorem), and we leave this as HW. $\square$

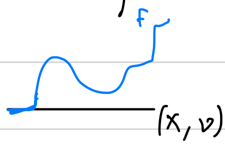
Def. Measure μ and ν on a measurable space $(X, \mathcal{B})$ are said to be *equivalent*, denoted $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$, i.e. they have the same null sets.

As a corollary from Lebesgue decomposition, we get:

Corollary. Let μ, ν be σ-finite measures on a measurable space $(X, \mathcal{B})$. Then there is a partition $X = X_0 \sqcup X_1$, $X_i \in \mathcal{B}$, such that $\mu|_{X_0} \sim \nu|_{X_0}$ and $\mu|_{X_1} \perp \nu|_{X_1}$. Such a partition is unique up to sets that are μ and ν null.

Proof. Applying Lebesgue decomp. to μ, ν, we get $\mu = \mu_0 + \mu_1$ with $\mu_0 \ll \nu$ and $\mu_1 \perp \nu$. Applying Lebesgue decomposition again to ν and μ_0, we get $\nu = \nu_0 + \nu_1$ such that $\nu_0 \ll \mu_0$ and $\nu_1 \perp \mu_0$, so $\mu_0 \sim \nu_0$. Now $\mu_1 \perp \nu$ yields a partition $Y = Y_0 \sqcup Y_1$, with Y_i in $\mathcal{B}$, such that $\mu_1(Y_0) = 0 = \nu(Y_1)$ so $\mu_0(Y_1) = 0$. Similarly, $\nu_1 \perp \mu_0$ yields a partition $Y_0 = Z_0 \sqcup Z_1$ such that $\mu_0(Z_1) = 0 = \nu_1(Z_0)$, so $\nu_0(Z_1) = 0$. Thus, $X_0 := Z_0$ and $X_1 = Z_1 \sqcup Y_1$ is as desired. The uniqueness is by checking that for any other such partition $X = X_0' \sqcup X_1'$, $\mu(X_i \cap X_{1-i}') = 0 = \nu(X_i \cap X_{1-i}')$ for $i = 0, 1$. $\square$

We would now like to understand the condition $\mu \ll \nu$ better. Given a measure ν, an example of such a μ is given by μ_f where f is a nonnegative measurable function:



The following theorem says that this is the only example for σ-finite measures.

Radon–Nikodym theorem. Let μ, ν be σ-finite measures on a measure space $(X, \mathcal{B})$. If $\mu \ll \nu$, then there is a $\mathcal{B}$-measurable non-negative function $f : X \to [0, \infty)$ such that $\mu = \nu_f$, i.e. $\mu(B) = \int_B f \, d\nu$ for all $B \in \mathcal{B}$.

To prove this theorem, it would be convenient to subtract one measure from another, which yields a so-called *signed measure*. We thus take a detour to define signed measures and prove a useful decomposition theorem for them, postponing the proof of the Radon–Nikodym theorem.

Signed measures.

Def. Let $(X, \mathcal{B})$ be a measurable space. A *signed measure* on $(X, \mathcal{B})$ is a function $\mu : \mathcal{B} \to \overline{\mathbb{R}} := [-\infty, \infty]$ such that

(i) $\mu(\emptyset) = 0$.

(ii) Ctbl-additivity: $\mu\left(\bigsqcup_{n \in \mathbb{N}} B_n\right) = \sum_{n \in \mathbb{N}} \mu(B_n)$.

(iii) μ doesn't attain both values $-\infty$ and $+\infty$.

Remark. It follows from condition (iii) that in condition (ii), either the positive or the negative terms of the series sum up to a finite number.

Examples.

(a) All measures are signed measures.

(b) If μ and ν are measures on $(X, \mathcal{B})$ and at least one of them is finite, then $\mu - \nu$ is a signed measure.

(c) For any $f \in L^1(X, \mu)$, $\mu_f := f\,d\mu$ is a signed measure, where

$$\mu_f(B) := \int_B f\,d\mu$$

for all μ-measurable $B \subseteq X$. In fact, it is a finite signed measure.

We now show that all signed measures are given in example (b).

Jordan Decomposition Theorem. Every signed measure ξ on a measurable space $(X, \mathcal{B})$ is equal to $\mu - \nu$ where μ, ν are measures on $(X, \mathcal{B})$ (thus at least one of μ and ν is finite).

This theorem follows from another decomposition theorem:

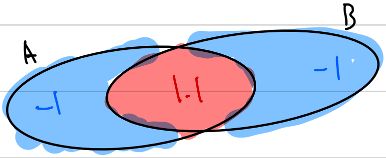
Hahn Decomposition Theorem. For any signed measure ξ on a measurable space $(X, \mathcal{B})$ there is a partition $X = X_+ \sqcup X_-$, $X_\pm \in \mathcal{B}$, such that $\xi|_{X_+}$ and $-\xi|_{X_-}$ are measures.

From this the Jordan decomposition follows by taking $\mu := \xi|_{X_+}$ and $\nu := -\xi|_{X_-}$.

i.e. $\mu(X_-) = 0 = \nu(X_+)$.

Def. For a signed measure ξ on $(X, \mathcal{B})$, call a set $B \in \mathcal{B}$ *purely positive* (resp. *purely negative*) if $\forall C \subseteq B, C \in \mathcal{B}$, $\xi(C) \geq 0$ (resp. $\xi(C) \leq 0$).

Caution. A union of two positive sets A, B (i.e. $\xi(A), \xi(B) \geq 0$) may not be positive:



$\xi(A \cup B) = -1 + 1.1 - 1 = -0.9 < 0$

while $\xi(A) = \xi(B) = 1.1 - 1 = 0.1 > 0$.

However, positive sets (as well as negative sets) are closed under disjoint unions.

Proof of Hahn decomposition. We need to partition X into purely positive and purely negative parts. Assume WLOG that $\xi < \infty$.

Claim. Every non-null positive set $P \subseteq X$ contains a non-null purely positive set $P_+ \subseteq P$ with $\xi(P_+) \geq \xi(P) > 0$.

Proof. We prove this by $\frac{1}{2}$-measure exhaustion. Let $N_0 \subseteq P$ be $\frac{1}{2}$-largest negative set, i.e.

$$-\xi(N_0) \geq \frac{1}{2} \overline{\sup} \{-\xi(N) : N \subseteq P \text{ and } N \text{ is negative}\},$$

where $\overline{\sup} := \min\{\sup, 1\}$. Then $P \setminus N_0$ is still positive (in fact, $\xi(P \setminus N_0) \geq \xi(P)$), so we repeat. Given $N_0, \dots, N_k$ pairwise disjoint negative subsets of P, we take as N_{k+1} a $\frac{1}{2}$-largest negative set in $P \setminus \bigsqcup_{i=0}^{k} N_i$, i.e.

$$-\xi(N_{k+1}) \geq \frac{1}{2} \sup\left\{ -\xi(N) : N \subseteq P \setminus \bigsqcup_{i=0}^{k} N_i \text{ and } N \text{ is negative} \right\}.$$

Let $N := \bigsqcup_{k \in \mathbb{N}} N_k$, so N is negative and $P_+ := P \setminus N$ is positive and $\xi(P_+) \geq \xi(P)$. We show that P_+ is purely positive. Indeed, because $\xi(P) - |\xi(N)| = \xi(P_+) < \infty$, it must be that $|\xi(N)| < \infty$, thus the sequence $(-\xi(N_k))_{k \in \mathbb{N}}$ is summable, so $\xi(N_k) \to 0$. If there were a non-null negative set $\tilde{N} \subseteq P_+$, then $\exists k$ for which $|\xi(N_k)| < \frac{1}{2}\xi(\tilde{N})$, contradicting the choice of N_k. □